Matrices and Systems of Equations

Section 7.1

College Algebra, MATH 171

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An "array" of numbers

- In this section, we express a system of equations as a rectangular array of numbers, called a ***matrix***.
  - One of them=*matrix*
  - More than one=*matrices*
- There are certain elements of a matrix that we define next.

An "*m-by-n*" matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

- An ***m× n matrix*** is a rectangular array of numbers with:
  - ***m rows*** and
  - ***n columns***
- We say that the matrix's dimension is *m× n* or that it is of ***order*** *m× n.*
- *Any number of the form* $\boldsymbol{a_{ij}}$ is called an ***entry*** of the matrix.
  - *The subscripts of the entry* $\boldsymbol{a_{ij}}$ *indicate it is in the* $\boldsymbol{i^{th}}$ ***row*** *and the* $\boldsymbol{j^{th}}$ ***column****.*

Augmented Matrix

- An ***augmented matrix*** is one that takes a system of linear equations and writes only its coefficients and constants in matrix form.
  - In the example below, notice that a missing variable in an equation corresponds to a 0 entry in the augmented matrix.
  - A ***coefficient matrix*** uses the coefficients of the system, but does ***not*** include the constant terms.

Linear System

$$\begin{cases} 3x - 2y + z = 5 \\ x + 3y - z = 0 \\ -x + 4z = 11 \end{cases}$$

Augmented Matrix

$$\begin{bmatrix} 3 & -2 & 1 & 5 \\ 1 & 3 & -1 & 0 \\ -1 & 0 & 4 & 11 \end{bmatrix}$$

Example 1

- Write the augmented matrix of the system of equations.

$$\begin{cases} 6x - 2y - \quad z = 4 \\ x + 3z = 1 \\ 7y + z = 5 \end{cases}$$

$$\begin{bmatrix} & & & \\ & & & \\ & & & \end{bmatrix}$$

Elementary Row Operations

- The operations used in Section 6.3 when solving systems of linear equations correspond to operations used on the rows of the augmented matrix of the system.
- They are:
  1. Adding a multiple of one row to another.
  2. Multiplying a row by a nonzero constant.
  3. Interchanging two rows.
- To keep track of the steps we use to solve a system, we can use some basic notation to help recall our steps.
  - Some examples are shown on page 529.

Example 2

- Solve the system of linear equations.

$$\begin{cases} x - y + 3z = 4 \\ x + 2y - 2z = 10 \\ 3x - y + 5z = 14 \end{cases}$$

Gaussian Elimination

- In general, to solve our system using its augmented matrix, we are looking for a matrix in a certain form, ***row-echelon form***.
- A matrix is in ***row-echelon form*** if it satisfies the following conditions.
  1. The first nonzero number in each row (read from left to right) is a 1. This is called the ***leading entry***.
  2. The leading entry in each row is to the right of the leading entry in the row immediately above it.
  3. All rows that are entirely zeros are at the bottom of the matrix.
- A matrix is in ***reduced row-echelon form*** if it is in row-echelon form and also meets this condition:
  4. Every number above and below each leading entry is a 0.

WTF: What's The Form?

- ***Reduced*** **row-echelon form**

$$\begin{bmatrix} 1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Leading 1's have 0's above and below them.

- **Row-echelon form**

$$\begin{bmatrix} 1 & 3 & -6 & 10 & 0 \\ 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Leading 1's shift to the right in successive rows.

- **NOT Row-echelon form**

$$\begin{bmatrix} 0 & 1 & -\frac{1}{2} & 0 & 7 \\ 1 & 0 & 3 & 4 & -5 \\ 0 & 0 & 0 & 1 & 0.4 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

Leading 1's ***do not*** shift to the right in successive rows.

Using Row Operations: A Standard Approach

- Start by working to get a 1 in the top left corner of your matrix. Then, obtain 0’s below that 1 by adding appropriate multiples of the first row to the rows below it.
- Next, obtain a leading 1 in the next row, and then obtain 0’s below that 1.
- At each stage make sure that every leading entry is to the right of the leading entry in the row above it--rearrange rows if necessary (you’re allowed to do it anyway).
- Continue this process until you arrive at a matrix in row-echelon form.

Gaussian elimination: What it looks like

- The technique described on the previous slide is called ***Gaussian elimination***, in honor of its inventor.
- For a 3 × 4 augmented matrix, here is what the process would look like.
  - The gray figures are arbitrary values in the matrix and will be used once back-substitution is possible.

$$\begin{bmatrix} 1 & \blacksquare & \blacksquare & \blacksquare \\ 0 & \blacksquare & \blacksquare & \blacksquare \\ 0 & \blacksquare & \blacksquare & \blacksquare \end{bmatrix} \rightarrow \begin{bmatrix} 1 & \blacksquare & \blacksquare & \blacksquare \\ 0 & 1 & \blacksquare & \blacksquare \\ 0 & 0 & \blacksquare & \blacksquare \end{bmatrix} \rightarrow \begin{bmatrix} 1 & \blacksquare & \blacksquare & \blacksquare \\ 0 & 1 & \blacksquare & \blacksquare \\ 0 & 0 & 1 & \blacksquare \end{bmatrix}$$

- In short, Gaussian elimination is:
  1. ***Augmented matrix.*** Write the augmented matrix for the system.
  2. ***Row-echelon form.*** Use elementary row operations to change the augmented matrix to row-echelon form.
  3. ***Back-substitution.*** Write the new system of equations that corresponds to the matrix and solve by back-substitution.

Example 3

- Solve the system of linear equations by using Gaussian elimination.

$$\begin{cases} 4x + 8y - 4z = 4 \\ 3x + 8y + 5z = -11 \\ -2x + y + 12z = -17 \end{cases}$$

Gauss-Jordan elimination: A good thing made better

- Gauss-Jordan elimination is very similar to Gaussian elimination, except our resulting matrix is in ***reduced*** row-echelon form instead of stopping at row-echelon form and back-substituting.
- To put a matrix in reduced row-echelon form:
  - Use the elementary row operations to put the matrix in row-echelon form.
  - Obtain zeros above each leading entry by adding multiples of the row with that leading entry to the rows above it. Begin with the last entry and work your way up.
- It will look like this:

$$\begin{bmatrix} 1 & \blacksquare & \blacksquare & \blacksquare \\ 0 & 1 & \blacksquare & \blacksquare \\ 0 & 0 & 1 & \blacksquare \end{bmatrix} \rightarrow \begin{bmatrix} 1 & \blacksquare & 0 & \blacksquare \\ 0 & 1 & 0 & \blacksquare \\ 0 & 0 & 1 & \blacksquare \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & \blacksquare \\ 0 & 1 & 0 & \blacksquare \\ 0 & 0 & 1 & \blacksquare \end{bmatrix}$$

Example 4 (actually Ex. 3, cont'd.)

- Solve the system of linear equations by using Gauss-Jordan elimination.

$$\begin{cases} 4x + 8y - 4z = 4 \\ 3x + 8y + 5z = -11 \\ -2x + y + 12z = -17 \end{cases}$$

Inconsistent and Dependent Systems

- From row-echelon form, one of the following conclusions must be true:

$$\begin{bmatrix} 1 & 2 & 5 & 7 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 6 & -1 & 3 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 1 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -3 & 1 \\ 0 & 1 & 5 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

  - **No Solution.** If the row-echelon form contains a row that represents the equation $0 = c$ where $c \neq 0$, then the system has no solution and is considered ***inconsistent.***
  - **One solution.** If each variable in the row-echelon form is a leading variable, then the system has exactly one solution, which can be found using back-substitution or Gauss-Jordan elimination.
  - **Infinitely many solutions.** If the variables in the row-echelon form are not all leading variables, and if the system is not inconsistent, then it has infinitely many solutions and is considered ***dependent***. Solve the system by reducing the matrix to reduced row-echelon form and then express the leading variables in terms of the nonleading variables. It is possible for the nonleading variables to take on real number values.

Assessment

Pgs. 534-536:
#'s 5 - 70, multiples of 5