

The Binomial Theorem



Section 9.6

College Algebra, MATH 171

Mr. Keltner

Binomials

- An expression in the form $\mathbf{a + b}$ is called a ***binomial***, because it is made of two unlike terms.
- We could use the FOIL method repeatedly to evaluate expressions like $(a + b)^2$, $(a + b)^3$, or $(a + b)^4$.
 - $(a + b)^2 = a^2 + 2ab + b^2$
 - $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
 - $(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$
- But to evaluate to higher powers of $(a + b)^n$ would be a difficult and tedious process.



Some Patterns to Notice

- For a binomial expansion of $(a + b)^n$, look at the expansions below:
 - $(a + b)^2 = a^2 + 2ab + b^2$
 - $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
 - $(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$
- Some simple patterns emerge by looking at these examples:
 - There are $n + 1$ terms, the first one is a^n and the last is b^n .
 - The exponent of a *decreases* by 1 for each term and the exponents of b *increase* by 1.
 - The sum of the exponents in each term is n .



For bigger exponents, how?

- To evaluate $(a + b)^8$, we will find a way to calculate the value of each coefficient.

$$(a + b)^8 =$$

$$a^8 + __ a^7b + __ a^6b^2 + __ a^5b^3 + __ a^4b^4 + __ a^3b^5 + __ a^2b^6 + __ ab^7 + b^8$$

- Pascal's Triangle will allow us to figure out what the coefficients of each term will be.
- The basic premise of Pascal's Triangle is that every entry (other than a 1) is the sum of the two entries diagonally above it.
 - ✦ We will show this more in depth later this section.
- This becomes useful in situations like our last example, or if we are only asked to identify a particular term of a binomial expansion.



The idea of a Factorial

- In any of the examples we had done already, notice that the coefficient of a^n and b^n were each 1.
 - Also, notice that the coefficient of a^{n-1} and a were each n .
- These values can be calculated by using ***factorials***.
 - ***n factorial*** is written as ***n!*** and calculated by multiplying the positive whole numbers less than or equal to n .
 - ✦ Formula: For $n \geq 1$, $n! = n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot 3 \cdot 2 \cdot 1$.
 - ✦ Example: $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$
 - Special cases: $0! = 1$ and $1! = 1$, to avoid division by zero in the next formula.



The Binomial Coefficient

- To find the coefficient of any term of $(a + b)^n$, we can apply factorials, using the formula:

$${}_n C_r = \binom{n}{r} = \frac{n!}{r! \cdot (n - r)!}$$

- where n is the power of the binomial expansion, $(a + b)^n$, and
- r is the exponent of b for the specific term we are calculating.
- So, for the second term of $(a + b)^8$, we would have $n = 8$ and $r = 1$ (because the second term is a^7b).
 - This procedure could be repeated for any term we choose, or all of the terms, one after another.
 - However, there is an easier way to calculate these coefficients.

Example 1

- Use the formula for the binomial coefficient to calculate each value

$${}_n C_r = \binom{n}{r} = \frac{n!}{r! \cdot (n-r)!}$$

$$\binom{9}{4}$$

$$\binom{100}{3}$$

$$\binom{100}{97}$$

$$\binom{13}{13}$$



Formulas to Numbers

- For $(a + b)^0$, we remember that anything to the zero power is 1.
 - This means that ${}_0C_0 = 1$, where $n=0$ and $r=0$.
 - Use this same reasoning to calculate ${}_1C_0$ and ${}_1C_1$, to find the coefficients of $(a + b)^1$.
 - What are those coefficients? $\underline{\quad}a + \underline{\quad}b$
- When finding a binomial's coefficients, we can only have values of r such that $0 \leq r \leq n$.



Finding Coefficients

- What we will notice is that when $r=0$ and when $r=n$, then ${}_nC_r=1$, no matter how big n becomes. This is because:

$${}_nC_0 = \frac{n!}{(n-0)! \cdot 0!} = \frac{n!}{n! \cdot 0!} = 1$$

$${}_nC_n = \frac{n!}{(n-n)! \cdot n!} = \frac{n!}{0! \cdot n!} = 1$$

- Note also that when $r = 1$ and $r = (n-1)$:

$${}_nC_1 = \frac{n!}{(n-1)! \cdot 1!} = \frac{n(n-1)!}{(n-1)! \cdot 1!} = n$$

$${}_nC_{n-1} = \frac{n!}{(n-(n-1))! \cdot (n-1)!} = \frac{n(n-1)!}{1! \cdot (n-1)!} = n$$

- *So, the coefficients of the first and last terms will always be 1.*

- *The **second coefficient** and **next-to-last coefficient** will be **n** . (because the denominators of their formulas are equal)*



Constructing Pascal's Triangle

- Continue evaluating ${}_n C_r$ for $n=2$ and $n=3$. When we include all the possible values of r such that $0 \leq r \leq n$, we get the figure below:

$n=0$	${}_0 C_0$
$n=1$	${}_1 C_0 \quad {}_1 C_1$
$n=2$	${}_2 C_0 \quad {}_2 C_1 \quad {}_2 C_2$
$n=3$	${}_3 C_0 \quad {}_3 C_1 \quad {}_3 C_2 \quad {}_3 C_3$
$n=4$	${}_4 C_0 \quad {}_4 C_1 \quad {}_4 C_2 \quad {}_4 C_3 \quad {}_4 C_4$
$n=5$	${}_5 C_0 \quad {}_5 C_1 \quad {}_5 C_2 \quad {}_5 C_3 \quad {}_5 C_4 \quad {}_5 C_5$
$n=6$	${}_6 C_0 \quad {}_6 C_1 \quad {}_6 C_2 \quad {}_6 C_3 \quad {}_6 C_4 \quad {}_6 C_5 \quad {}_6 C_6$



Constructing Pascal's Triangle

- Knowing what we know about ${}_n C_r$ and its values when $r=0$, 1 , $(n-1)$, and n , we can fill out the outside values of the Triangle:

$$r=0, {}_n C_r=1$$

$$r=n, {}_n C_r=1$$

$$r=1, {}_n C_r=n$$

$$r=(n-1), {}_n C_r=n$$

$n=0$	1
$n=1$	$1 \ 1$
$n=2$	$1 \ 2 \ 1$
$n=3$	$1 \ 3 \ 3 \ 1$
$n=4$	$1 \ 4 \ 6C_2 \ 4 \ 1$
$n=5$	$1 \ 5 \ 10C_2 \ 10C_3 \ 5 \ 1$
$n=6$	$1 \ 6 \ 15C_2 \ 20C_3 \ 15C_4 \ 6 \ 1$



Sum thing from above

- To complete Pascal's Triangle, note that each row entry that is NOT on the outer diagonal is just the sum of the two numbers above it. Use this idea to fill in the missing values

$n=0$	1
$n=1$	$1 \quad 1$
$n=2$	$1 \quad 2 \quad 1$
$n=3$	$1 \quad 3 \quad 3 \quad 1$
$n=4$	$1 \quad 4 \quad \quad 4 \quad 1$
$n=5$	$1 \quad 5 \quad \quad \quad 5 \quad 1$
$n=6$	$1 \quad 6 \quad \quad \quad \quad 6 \quad 1$



Using Pascal's Triangle

- We can also use Pascal's Triangle to expand binomials, such as $(x - 3)^4$.
- The numbers in Pascal's Triangle can be used to find the coefficients in a binomial expansion.
- For example, the coefficients in $(x - 3)^4$ are represented by the row of Pascal's Triangle for $n = 4$.

$$\begin{aligned}(x - 3)^4 &= {}_4C_0(x)^4(-3)^0 + {}_4C_1(x)^3(-3)^1 + {}_4C_2(x)^2(-3)^2 + {}_4C_3(x)^1(-3)^3 + {}_4C_4(x)^0(-3)^4 \\ &= 1(x)^4(1) + 4(x)^3(-3) + 6(x)^2(9) + 4(x)^1(-27) + 1(x)^0(81) \\ &= 1x^4 - 12x^3 + 54x^2 - 108x + 81\end{aligned}$$

Pascal's Triangle

- What your textbook emphasizes is that *every entry (other than a 1) is the sum of the two entries diagonally above it.*
- Back to our example of expanding $(a + b)^8$.

$n=0$	1
$n=1$	1 1
$n=2$	1 2 1
$n=3$	1 3 3 1
$n=4$	1 4 6 4 1
$n=5$	1 5 10 10 5 1
$n=6$	1 6 15 20 15 6 1
$n=7$	
$n=8$	

Example 3

- Find the expansion of $(a + b)^7$ by using Pascal's triangle.

$n=0$	1
$n=1$	1 1
$n=2$	1 2 1
$n=3$	1 3 3 1
$n=4$	1 4 6 4 1
$n=5$	1 5 10 10 5 1
$n=6$	1 6 15 20 15 6 1
$n=7$	



The Binomial Theorem

- The Binomial Theorem states what we have already used in the past two examples, just with a lot more notation involved.
- The general idea of the Binomial Theorem is that:
 - The term that contains a^r in the expansion $(a + b)^n$ is

$$\binom{n}{n-r} a^r b^{n-r} \quad \text{or} \quad \frac{n!}{(n-r)! \cdot r!} a^r b^{n-r}$$

- It helps to remember that the sum of the exponents of each term of the expansion is n . (In our formula, note that $r + (n - r) = n$.)
- **Example 4:** Find the term that contains x^5 in the expansion of $(2x + y)^{20}$.



The Binomial Theorem

- **Example 4:** Find the term that contains x^8 in the expansion of $\left(x^2 + \frac{1}{x}\right)^{10}$.
 - It may help to find a general way of expressing each term of the expression, since both of the terms could be considered powers of x .

$$\binom{n}{n-r} a^r b^{n-r}$$

$$\frac{n!}{(n-r)! \cdot r!} a^r b^{n-r}$$

Assessment



**Pg. 646:
#’s 7-42, multiples of 7**